On the Phase Diagram of Josephson Junction Arrays with Offset Charges*

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We study the effects of external offset charges on the phase diagram of Josephson junction arrays. Using the path integral approach, we provide a pedagogical derivation of the equation for the phase boundary line between the insulating and the superconducting phase within the mean-field theory approximation. For a uniform offset charge $q = e$ the superconducting phase increases with respect to $q = 0$ and a characteristic lobe structure appears in the phase diagram when the critical line is plotted as a function of q at fixed temperature. We review our analysis of the physically relevant situation where a Josephson network feels the effect of random offset charges. We observe that the Mott-insulating lobe structure of the phase diagram disappears for large variance ($\sigma \gtrsim e$) of the offset charges probability distribution; with nearest-neighbor interactions, the insulating lobe around $q = e$ is destroyed even for small values of σ. Finally, we study the case of random self-capacitances: here we observe that, until the variance of the distribution reaches a critical value, the superconducting phase increases in comparison to the situation in which all self-capacitances are equal.

I. INTRODUCTION

Josephson junction arrays (JJA) are an ideal model to study a variety of phenomena such as phase transitions, frustration effects, and vortex dynamics [1]. The high level of accuracy reached in their experimental realization makes JJA relevant for different applications, ranging from possible implementations of quantum computation schemes [2] to the realization of topological order [3]. Furthermore, particular geometries of JJA may induce novel classical coherent states [4].

JJA and granular superconductors, namely systems of metallic grains embedded in an insulator, become superconducting in two steps [5]. First, at the bulk critical temperature, each grain develops a superconducting gap but the phases of the order parameter on different grains are uncorrelated. Then, at a lower temperature T_c, the Cooper pair tunneling between grains gives rise to a long-range phase coherence and the system as a whole exhibits a phase transition to a superconducting state. The latter transition is governed by the competition between the Josephson tunneling, characterized by a Josephson coupling energy E_J, and the Coulomb interaction between Cooper pairs, described by a charging energy E_C. In classical junction arrays the Josephson coupling E_J is dominant and the transition separates a superconducting low temperature phase from a normal high temperature phase. When E_C is comparable to E_J (small grains), charging effects give rise to a quantum dynamics and the energy cost of Cooper pair tunneling may be higher then the energy gained by the formation of a phase-coherent state.

It is relevant to analyze in detail the effect of a background of external charges on the superconductor-insulator transition of a quantum JJA. Offset charges arise in real physical systems as a result of charged impurities or by application of a gate voltage between the array and the ground. In the former case, offset charges are distributed randomly on the lattice while in the latter case they play the role of a sort of chemical potential and their distribution can be uniform. Thus, offset charges may be regarded as effective charges $q_{\mathbf{i}}$, located at the sites of the lattice. When $q_{\mathbf{i}} \neq 2e$, the offset charges cannot be eliminated by Cooper pair tunneling. As it will be clear, offset charges frustrate the attempts of the system to minimize the energy of the charge distribution of the ground state (for this reason they are also called *frustration charges*). A large number of studies has by now been devoted to the analysis of the effects induced by offset charges on the (zero-temperature) quantum phase transition [6–9] and on the phase transition at finite T [10–12].

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The Hamiltonian commonly used to describe the Cooper pair tunneling in superconducting quantum networks defines the so called quantum phase model (QPM). In its most general form it is given by

$$H = \frac{1}{2}\sum_{ij}(Q_{\mathbf{i}} + q_{\mathbf{i}})C^{-1}_{\mathbf{ij}}(Q_{\mathbf{j}} + q_{\mathbf{j}}) - E_J \sum_{\langle ij \rangle}\cos(\varphi_{\mathbf{i}} - \varphi_{\mathbf{j}}) \tag{1}$$

where $\varphi_{\mathbf{i}}$ is the phase of the superconducting order parameter at the grain $\mathbf{i}$. Its conjugate variable $n_{\mathbf{i}}$ ($[\varphi_{\mathbf{i}}, n_{\mathbf{i}}] = i\,\delta_{\mathbf{ij}}$) describes the number of Cooper pairs in the $\mathbf{i}$-th superconducting grain. The symbol $\langle ij \rangle$ indicates a sum over nearest-neighbor grains only. The first term in the Hamiltonian (1) determines the electrostatic coupling between the Cooper pairs: $Q_{\mathbf{i}}$ is the excess of charge due to Cooper pairs ($Q_{\mathbf{i}} = 2en_{\mathbf{i}}$) on the site $\mathbf{i}$ and $C_{\mathbf{ij}}$ is the capacitance matrix. The diagonal elements of the inverse matrix $C^{-1}_{\mathbf{ij}}$ provide the charging energy: $E_C = e^2 C^{-1}_{\mathbf{ii}}/2 \equiv e^2/2C_0$, where C_0 is the self-capacitance. The second term describes the hopping of Cooper pairs between neighboring sites (E_J is the Josephson energy). The parameter $\alpha = zE_J/4E_C$ (where z is the coordination number) governs the superconductor-insulator transition: for $\alpha \ll 1$ the Josephson network is in the Mott-insulator state, while for $\alpha \gg 1$ is in the superconducting one. An external gate voltage $V_{\mathbf{i}}$ contributes to the energy via the offset charge $q_{\mathbf{i}} = \sum_{\mathbf{j}} C_{\mathbf{ij}} V_{\mathbf{j}}$. This external voltage can be either applied to the ground plane or, more realistically, it may be induced by charges trapped in the substrate. In the latter situation $q_{\mathbf{i}}$ is naturally a random variable: we shall review our study of this physically interesting situation in Section IV.

As it is well known [1], the QPM (1) is equivalent to the boson Hubbard model (BHM) in the limit of large particle numbers per junction. The BHM describes soft-core bosons hopping on a lattice [13] and is defined as

$$H = \frac{1}{2}\sum_{ij} n_{\mathbf{i}} U_{ij} n_{\mathbf{j}} - \mu \sum_{i} n_{\mathbf{i}} - t\sum_{\langle ij \rangle}(b^{\dagger}_{\mathbf{i}} b_{\mathbf{j}} + H.C.). \tag{2}$$

Here, $b^{\dagger}_{\mathbf{i}}$ ($b_{\mathbf{i}}$) is the creation (annihilation) operator for bosons and $n_{\mathbf{i}} = b^{\dagger}_{\mathbf{i}} b_{\mathbf{i}}$ is the number operator. By writing the $b_{\mathbf{i}}$'s in terms of their amplitudes and phases, and by neglecting the deviations of the amplitudes from the average number of Cooper pairs $\langle n \rangle$, we are lead to the QPM (1). An exact mapping between the two models has been derived in Ref. [14]. The hopping term is associated with the Josephson tunneling ($2\langle n \rangle t \to E_J$) whereas $U_{\mathbf{ij}} \to 4e^2 C^{-1}_{\mathbf{ij}}$ describes the Coulomb interactions between bosons. The chemical potential in the BHM plays a role analogous to the external charge in the QPM ($\mu \to q_{\mathbf{i}}$). Thus, a QPM with random offset charges corresponds to a BHM with random on-site energies.

In this paper we shall use a pedagogical approach to review the results obtained in our analysis [12,15] of the finite temperature superconductor-insulator transition in JJA. Using the finite temperature path integral approach, we provide an explicit derivation of the equation for the phase boundary for quantum JJA with offset charges and general capacitance matrix within the mean-field theory approximation [12,16]. Offset charges dramatically change the phase boundary: as it may be easily seen from Figs. 1 and 2, already with diagonal capacitance matrices, and for a uniform offset charge $q = e$, the superconducting phase increases with respect to the unfrustrated case. A lobe structure appears in the phase diagram when, at fixed temperature, the critical line is plotted against the offset charge q. In Fig. 3 we plot the value of α at which the transition occurs as a function of q for two different critical temperatures T_c. As it is shown in Fig. 3, the critical line is $2e$-periodic, and has a minimum in $q = e$, corresponding to the fact that for this value of q the superconducting region is maximal.

Within the same mean-field approach, we shall also treat JJA with capacitive disorder [15], i.e., random offset charges and/or random self-capacitances. This is motivated by the fact that, in practical realizations of Josephson devices, one has to deal with capacitance disorder caused either by offset charge defects in the junctions or in the substrate (random offset charges) [17] or by imperfections in the construction of the devices, which may lead to random capacitances of the Josephson junctions. Although from a theoretical point of view charge and magnetic frustration are expected to be dual to each other, experimentally it is possible to tune only the magnetic frustration in a controlled way. For this reason, it is widely believed that a challenging task for the theory is to develop reliable techniques to investigate the effects of random offset charges and capacitances on the phase structure of JJA.

Random offset charges could be forced to vanish by using a gate for each superconducting island; however, this procedure works only for small networks, since in large arrays too many gate would be necessary, making impossible the cooling of the system at the desired temperatures. In Ref. [18] the case of uniform charge frustration was analyzed experimentally. It was observed a sensible variation ($\sim 40\%$) of the resistance between the unfrustrated and the fully frustrated array: however, it was impossible to quench the Coulomb blockade as it can be done in small arrays.

During the last decade, much attention have been devoted to the Bose-Hubbard model with random on-site energies at $T = 0$ [13,19–24]. As it has been already pointed out, the properties of the disordered BHM are closely related to those of JJA with random offset charges. Without disorder, the BHM exhibits two types of phases: a superfluid

phase and a Mott insulating phase. Strong disorder may lead to the existence of a third, intermediate gapless phase exhibiting an infinite superfluid susceptibility and finite compressibility: the Bose glass (BG). It is by now known that, unless one introduces an *ad hoc* probability distribution, there is no BG in mean-field (MF) at $T = 0$ [13]; the reason why - at $T = 0$ - there is no BG phase in MF is better understood if one considers the infinite-range hopping limit of the Hamiltonian (2): $t_{ij} = t/N$ for all the sites $\mathbf{i} \neq \mathbf{j}$, N being the total number of sites. At $T = 0$, when $t = 0$ and $\langle n_{\mathbf{i}} \rangle = 0$ the system is, of course, in the insulating phase. If a small positive t is turned on, the infinite-ranged hopping allows the system to gain kinetic energy (with zero cost in the on-site potential energy) by moving bosons in unoccupied sites with the highest on-site energies; thus, the bosons are delocalized and the system becomes superfluid even for very small values of t. Of course, this argument does not hold at finite T; so far, there is no evidence for a Bose glass phase at finite temperature.

In the following we shall review the results of Ref. [15], where the finite temperature phase diagrams of JJA with random offset charges and/or random self-capacitances have been obtained.

The plan of the paper is the following: in Section II we use the coarse grained approach to compute the Ginzburg-Landau free energy for quantum JJA with charge frustration and a general Coulomb interaction matrix. The path integral providing the phase correlator needed to investigate the finite temperature critical behavior of the system is explicitly computed.

In Section III, within the mean field theory approximation, we derive the analytical form of the critical line equation from the Ginzburg-Landau free energy. The phase diagram is drawn in the diagonal case for a generic uniform offset charge distribution. We then analyze the low temperature limit of a system with nearest neighbor interaction matrix when a uniform background of external charges $q_{\mathbf{i}} = e$ is considered.

In Section IV we consider JJA with capacitive disorder (random offset charges and/or random self-capacitances): using the results of the previous Sections, we compute the free energy averaged over the disorder. Section IV.A is devoted to the study of the effects of random offset charges on the finite temperature phase diagram, with diagonal and nearest-neighbor capacitance matrices, while in Section IV.B we consider JJA with random self-capacitance.

Finally, Section V is devoted to concluding remarks.

II. PATH INTEGRAL APPROACH: THE GINZBURG-LANDAU FREE ENERGY

The partition function for the frustrated model described by the Hamiltonian (1) is given by

$$Z = \mathrm{Tr} e^{-\beta H} = \sum_n \left\langle \psi_n \left| e^{-\beta H} \right| \psi_n \right\rangle \tag{3}$$

where $\beta = 1/k_B T$ and the sum is extended only to states of charge $2e$ and thus with definite periodicity.

In the functional approach Z reads

$$Z = \int \prod_{\mathbf{i}} D\varphi_{\mathbf{i}} \exp\left\{ - \int_0^\beta d\tau L_E \left(\varphi_{\mathbf{i}}(\tau), \frac{d\varphi_{\mathbf{i}}}{d\tau}(\tau) \right) \right\} \tag{4}$$

where the Euclidean Lagrangian L_E may be derived from

$$L = \frac{1}{2}\left(\frac{\hbar}{2e}\right)^2 \sum_{\mathbf{ij}} C_{\mathbf{ij}} \frac{d\varphi_{\mathbf{i}}}{dt}\frac{d\varphi_{\mathbf{j}}}{dt} - \left(\frac{\hbar}{2e}\right) \sum_{\mathbf{i}} \frac{d\varphi_{\mathbf{i}}}{dt} q_{\mathbf{i}} + E_J \sum_{\langle \mathbf{ij} \rangle} \cos(\varphi_{\mathbf{i}} - \varphi_{\mathbf{j}}) \tag{5}$$

by replacing $it/\hbar \to \tau$. The path integral that one should compute is then given by:

$$Z = \int \prod_{\mathbf{i}} D\varphi_{\mathbf{i}} \;\; \exp\left\{ \int_0^\beta d\tau \big[- \frac{1}{2} \sum_{\mathbf{ij}} C_{\mathbf{ij}} \frac{\dot{\varphi}_{\mathbf{i}}}{2e}\frac{\dot{\varphi}_{\mathbf{j}}}{2e} + i \sum_{\mathbf{i}} q_{\mathbf{i}} \frac{\dot{\varphi}_{\mathbf{i}}}{2e} + \frac{E_J}{2} \sum_{\mathbf{i}\,\mathbf{j}} e^{i\varphi_{\mathbf{i}}} \gamma_{\mathbf{ij}} e^{-i\varphi_{\mathbf{j}}} \big] \right\} \tag{6}$$

where $-\infty < \varphi_{\mathbf{i}} < +\infty$, $\varphi_{\mathbf{i}}(0) = \varphi_{\mathbf{i}}(\beta) + 2\pi n_{\mathbf{i}}$ and $\gamma_{\mathbf{ij}} = 1$ if $\mathbf{i}, \mathbf{j}$ are nearest neighbors and equals zero otherwise (i.e., hopping term just between nearest neighbors). If $\gamma_{\mathbf{ij}} = 1$ for all pairs $\mathbf{i}, \mathbf{j}$ on the lattice, one is lead to the infinite-range hopping limit which provides a remarkable example of an exactly solvable MF theory [13]. The integers $n_{\mathbf{i}}$ appearing in the boundary conditions take into account the 2π-periodicity of the states ψ_n contributing to Eq. (3).

In order to derive the Ginzburg-Landau free energy for the order parameter, it is convenient to carry out the integration over the phase variables by means of the Hubbard-Stratonovich procedure [25]: using the identity

$$e^{J^+\Gamma J} = \frac{\det \Gamma^{-1}}{\pi^N} \int \prod_{\mathbf{i}} D^2\psi_{\mathbf{i}} \; e^{-\psi^+\Gamma^{-1}\psi - J^+\psi - \psi^+ J} \tag{7}$$

the partition function may be rewritten as

$$Z = \int \prod_{\mathbf{i}} D\psi_{\mathbf{i}} D\psi^*_{\mathbf{i}} e^{\int_0^\beta d\tau (-\frac{2}{E_J}\sum_{\mathbf{ij}} \psi^*_{\mathbf{i}} \gamma^{-1}_{\mathbf{ij}} \psi_{\mathbf{j}})} e^{-S_{Eff}[\psi]}, \tag{8}$$

where the effective action for the auxiliary Hubbard-Stratonovich field $\psi_{\mathbf{i}}$, $S_{Eff}[\psi]$, is given by

$$S_{Eff}[\psi] = -\log\Big\{ \int \prod_{\mathbf{i}} D\varphi_{\mathbf{i}} \; \exp\Big\{ \int_0^\beta d\tau \big[-\frac{1}{2}\sum_{\mathbf{ij}} C_{\mathbf{ij}} \frac{\dot\varphi_{\mathbf{i}}}{2e}\frac{\dot\varphi_{\mathbf{j}}}{2e} + i\sum_{\mathbf{i}} \big(q_{\mathbf{i}} \frac{\dot\varphi_{\mathbf{i}}}{2e} - \psi_{\mathbf{i}} e^{i\varphi_{\mathbf{i}}} - \psi^*_{\mathbf{i}} e^{-i\varphi_{\mathbf{i}}} \big) \big] \Big\} \Big\}. \tag{9}$$

The Hubbard-Stratonovich field $\psi_{\mathbf{i}}$ may be regarded as the order parameter for the insulator-superconductor phase transition since it turns out to be proportional to $\langle e^{i\varphi_{\mathbf{i}}} \rangle$, as it can be easily seen from the classical equations of motion. From Eq. (9), the Ginzburg-Landau free-energy may be derived by integrating out the phase field $\phi_{\mathbf{i}}$.

Since the superconductor-insulator phase transition is expected to be second order [26], it may be safely assumed that, close to the onset of superconductivity, the order parameter $\psi_{\mathbf{i}}$ is small. One may then expand the effective action up to the second order in $\psi_{\mathbf{i}}$, getting

$$S_{Eff}[\psi] = S_{Eff}[0] + \int_0^\beta d\tau \int_0^\beta d\tau' G_{\mathbf{rs}}(\tau,\tau') \psi_{\mathbf{r}}(\tau) \psi^*_{\mathbf{s}}(\tau') + \cdots \quad , \tag{10}$$

where $G_{\mathbf{rs}}$ is the phase correlator

$$G_{\mathbf{rs}}(\tau,\tau') = \frac{1}{\beta^2} \left. \frac{\delta^2 S_{Eff}[\psi]}{\delta\psi_{\mathbf{r}}(\tau)\delta\psi_{\mathbf{s}}(\tau')} \right|_{\psi,\psi^*=0} = \langle e^{i\varphi_{\mathbf{r}}(\tau) - i\varphi_{\mathbf{s}}(\tau')} \rangle_0. \tag{11}$$

The partition function (8) can then be written as [12]

$$Z = \int \prod_{\mathbf{i}} D\psi_{\mathbf{i}} D\psi^*_{\mathbf{i}} e^{-\beta F[\psi]}, \tag{12}$$

where $F[\psi]$ is the Ginzburg-Landau free energy; due to Eq. (10), up to the second order in $\psi_{\mathbf{i}}$, one has

$$F[\psi] = \int_0^\beta d\tau \int_0^\beta d\tau' \sum_{\mathbf{ij}} \psi^*_{\mathbf{i}}(\tau) [\gamma^{-1}_{\mathbf{ij}} \delta(\tau-\tau') - G_{\mathbf{ij}}(\tau,\tau')] \psi_{\mathbf{j}}(\tau'). \tag{13}$$

In order to compute the phase correlator $G_{\mathbf{rs}}$ one should evaluate the expectation value in Eq. (12) by means of the path integral over the phase variables $\varphi_{\mathbf{i}}(\tau)$. In performing the integration needed for the explicit evaluation of Eq. (13), one should take into account that the field configurations satisfy

$$\varphi_{\mathbf{i}}(\beta) - \varphi_{\mathbf{i}}(0) = 2\pi n_{\mathbf{i}}. \tag{14}$$

It turns out very convenient to untwist the boundary conditions by decomposing the phase field in terms of a periodic field $\phi_{\mathbf{i}}(\tau)$ and a term linear in τ which takes into account the boundary conditions (14); namely, one sets

$$\varphi_{\mathbf{i}}(\tau) = \phi_{\mathbf{i}}(\tau) + \frac{2\pi}{\beta} n_{\mathbf{i}} \tau \ , \tag{15}$$

with $\phi_{\mathbf{i}}(\beta) = \phi_{\mathbf{i}}(0)$. Summing over all the phases $\varphi_{\mathbf{i}}(\tau)$ amounts then to integrate over the periodic field $\phi_{\mathbf{i}}$ and to sum over the integers $n_{\mathbf{i}}$. As a result [12], the phase correlator factorizes as the product of a topological term depending on the integers $n_{\mathbf{i}}$ and a nontopological one; namely,

$$G_{\mathbf{rs}}(\tau;\tau') = \frac{\int D\phi_{\mathbf{i}} e^{i\phi_{\mathbf{r}}(\tau) - i\phi_{\mathbf{s}}(\tau')} \exp\Big\{ \int_0^\beta d\tau (-\frac{1}{2} C_{\mathbf{ij}} \frac{\dot\phi_{\mathbf{i}}}{2e}\frac{\dot\phi_{\mathbf{j}}}{2e}) \Big\}}{\int D\phi_{\mathbf{i}} \exp\Big\{ \int_0^\beta d\tau (-\frac{1}{2} C_{\mathbf{ij}} \frac{\dot\phi_{\mathbf{i}}}{2e}\frac{\dot\phi_{\mathbf{j}}}{2e}) \Big\}}.$$

$$\cdot\ \frac{\sum_{[n_{\mathbf{i}}]} e^{i\frac{2\pi}{\beta}(n_{\mathbf{r}}\tau - n_{\mathbf{s}}\tau')} e^{\left\{\sum_{\mathbf{ij}} -\frac{\pi^2}{2\beta e^2} C_{\mathbf{ij}} n_{\mathbf{i}} n_{\mathbf{j}} + \sum_{\mathbf{i}} 2i\pi \frac{q_{\mathbf{i}}}{2e} n_{\mathbf{i}}\right\}}}{\sum_{[n_{\mathbf{i}}]} e^{\left\{\sum_{\mathbf{ij}} -\frac{\pi^2}{2\beta e^2} C_{\mathbf{ij}} n_{\mathbf{i}} n_{\mathbf{j}} + \sum_{\mathbf{i}} 2i\pi\beta \frac{q_{\mathbf{i}}}{2e} n_{\mathbf{i}}\right\}}}\ . \tag{16}$$

For the sake of clarity, we briefly outline the computation of the factors appearing in the right-hand side of the previous equation. Firstly, one should compute the path integral

$$\frac{\int D\phi_{\mathbf{i}} e^{i\phi_{\mathbf{r}}(\tau) - i\phi_{\mathbf{s}}(\tau')} \exp\left\{\int_0^\beta d\tau (-\frac{1}{2} C_{\mathbf{ij}} \frac{\dot{\phi}_{\mathbf{i}}}{2e} \frac{\dot{\phi}_{\mathbf{j}}}{2e})\right\}}{\int D\phi_{\mathbf{i}} \exp\left\{\int_0^\beta d\tau (-\frac{1}{2} C_{\mathbf{ij}} \frac{\dot{\phi}_{\mathbf{i}}}{2e} \frac{\dot{\phi}_{\mathbf{j}}}{2e})\right\}}\ . \tag{17}$$

Fourier transforming $\phi_{\mathbf{i}}(\tau)$ according to

$$\phi_{\mathbf{i}}(\tau) = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \phi_{\mathbf{i},m} e^{i\omega_m \tau} \tag{18}$$

with $0 \leq \tau \leq \beta$ and $\omega_m = \frac{2\pi}{\beta} m$, the numerator of Eq. (17) becomes

$$\int \prod_{\mathbf{i}} d\phi_{\mathbf{i},0} \prod_{n=1}^{\infty} d\phi_{\mathbf{i},n} d\phi^*_{\mathbf{i},n} \exp\left\{ - \frac{1}{4e^2\beta} \sum_{\mathbf{ij}} \sum_{n=1}^{+\infty} C_{\mathbf{ij}} \omega_n^2 \phi_{\mathbf{i},n} \phi^*_{\mathbf{j},n}\right.$$

$$\left. + \frac{i}{\beta} \sum_{n=1}^{\infty} \left(\phi_{\mathbf{r},n} e^{i\omega_n \tau} - \phi^*_{\mathbf{s},n} e^{-i\omega_n \tau'} \right) + \frac{i}{\beta} \left(\phi_{\mathbf{r},0} - \phi_{\mathbf{s},0}\right) + c.c. \right\}\ . \tag{19}$$

Upon integrating over the components $\phi_{\mathbf{r},0}$, $\phi_{\mathbf{s}0}$ one gets a factor $\delta_{\mathbf{rs}}$

$$\left(\prod_{\mathbf{i} \neq \mathbf{r},\mathbf{s}} \int_{-\infty}^{\infty} d\phi_{\mathbf{i},0} \right) \left(\int_{-\infty}^{\infty} d\phi_{\mathbf{r},0} \int_{-\infty}^{\infty} d\phi_{\mathbf{s},0} e^{\frac{i}{\beta}(\phi_{\mathbf{r},0} - \phi_{\mathbf{s},0})} \right) = \delta_{\mathbf{rs}} \cdot K \tag{20}$$

where K is an irrelevant divergent constant which cancels against the denominator. Using Eq. (20), Eq. (19) becomes

$$K\delta_{\mathbf{rs}} \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} \prod_{\mathbf{i}} d\phi_{\mathbf{i},n} d\phi^*_{\mathbf{i},n} \exp\left\{ - \frac{1}{4e^2\beta} \sum_{\mathbf{ij}} C_{\mathbf{ij}} \omega_n^2 \phi_{\mathbf{i},n} \phi^*_{\mathbf{j},n}\right.$$

$$\left. + \sum_{\mathbf{i}} \phi_{\mathbf{i},n} \frac{i}{\beta} \delta_{\mathbf{ri}} (e^{i\omega_n \tau} - e^{i\omega_n \tau'}) - \sum_{\mathbf{i}} \phi^*_{\mathbf{i},n} \delta_{\mathbf{ri}} \frac{i}{\beta} (e^{-i\omega_n \tau'} - e^{-i\omega_n \tau'}) \right\}.$$

The multiple Gaussian integral is easily computed to give [12], up to an irrelevant constant which cancels against the denominator,

$$\delta_{\mathbf{rs}} \prod_{\mathbf{i}} \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} d\phi_{\mathbf{i}n} d\phi^*_{\mathbf{i}n} \exp\left\{ \sum_{\mathbf{ij}} \frac{i}{\beta} \delta_{\mathbf{ri}} (e^{i\omega_n \tau} - e^{i\omega_n \tau'}) \cdot\right.$$

$$\left. \cdot (\frac{4e^2 \beta C^{-1}_{\mathbf{ij}}}{\omega_n^2}) \frac{i}{\beta} \delta_{\mathbf{ri}} (e^{-i\omega_n \tau} - e^{-i\omega_n \tau'}) \right\} =$$

$$= \delta_{\mathbf{rs}} \exp\left\{ \frac{8e^2 C^{-1}_{\mathbf{rr}}}{\beta} \sum_{n=1}^{\infty} (\frac{1 - \cos\omega_n(\tau - \tau')}{\omega_n^2}) \right\}$$

where $-\beta \leq \tau - \tau' \leq \beta$. By using the identity

$$|x| - \frac{x^2}{\beta} = \sum_{n=1}^{\infty} \left(\frac{4}{\beta \omega_n^2} - \frac{4 \cos \omega_n x}{\beta \omega_n^2} \right) \qquad - \beta \leq x \leq \beta, \tag{21}$$

the first term of the right-hand side of Eq. (16) is found to be

$$\delta_{\mathbf{rs}} \exp\left\{ -2e^2 C^{-1}_{\mathbf{rr}} \left(|\tau - \tau'| - \frac{(\tau - \tau')^2}{\beta} \right) \right\}. \tag{22}$$

The sum over the integers in the topological factor in Eq. (16) is carried out by means of the Poisson resummation formula [27]

$$|\det G|^{\frac{1}{2}} \sum_{[n_{\mathbf{i}}]} e^{-\pi (n-a)_{\mathbf{i}} G_{\mathbf{ij}} (n-a)_{\mathbf{j}}} = \sum_{[m_{\mathbf{i}}]} e^{-\pi m_{\mathbf{i}} (G^{-1})_{\mathbf{ij}} m_{\mathbf{j}} - 2\pi i m_{\mathbf{i}} a_{\mathbf{i}}} \ . \tag{23}$$

Due to Eq. (23), Eq. (16) becomes

$$G_{\mathbf{rs}}(\tau, \tau') = \delta_{\mathbf{rs}} e^{-2e^2 C^{-1}_{\mathbf{rr}} |\tau - \tau'|} \ .$$

$$\cdot \frac{\sum_{[n_{\mathbf{i}}]} e^{-\sum_{\mathbf{ij}} 2e^2 \beta C^{-1}_{\mathbf{ij}} (n_{\mathbf{i}} + \frac{q_{\mathbf{i}}}{2e})(n_{\mathbf{j}} + \frac{q_{\mathbf{j}}}{2e}) - \sum_{\mathbf{i}} 4e^2 C^{-1}_{\mathbf{ri}} (n_{\mathbf{i}} + \frac{q_{\mathbf{i}}}{2e})(\tau - \tau')}}{\sum_{[n_{\mathbf{i}}]} e^{\sum_{\mathbf{ij}} 2\beta e^2 C^{-1}_{\mathbf{ij}} (n_{\mathbf{i}} + \frac{q_{\mathbf{i}}}{2e})(n_{\mathbf{j}} + \frac{q_{\mathbf{j}}}{2e})}} \tag{24}$$

with $n_{\mathbf{i}}$ taking all integer values and $\sum_{[n_{\mathbf{i}}]}$ being a sum over all the configurations.

By means of a Euclidean-time Fourier transform, the fields $\psi_{\mathbf{i}}$ are defined as

$$\psi_{\mathbf{i}}(\tau) = \frac{1}{\beta} \sum_{\mu} \psi_{\mathbf{i}}(\omega_\mu) e^{i \omega_\mu \tau}$$

where ω_μ are the Matsubara frequencies. As a consequence, the phase correlator $G_{\mathbf{ij}}$ may be written as

$$G_{\mathbf{ij}}(\tau; \tau') = \frac{1}{\beta} \sum_{\mu \mu'} G_{\mathbf{ij}}(\omega_\mu; \omega_{\mu'}) e^{i \omega_\mu \tau} e^{i \omega_{\mu'} \tau'} \ . \tag{25}$$

From Eq. (24) one can easily show that $G_{\mathbf{rs}}(\omega_\mu; \omega'_\mu)$ is diagonal in the Matsubara frequencies and is given by

$$G_{\mathbf{rs}}(\omega_\mu; \omega'_\mu) = G_{\mathbf{r}}(\omega_\mu) \cdot \delta_{\mathbf{rs}} \cdot \delta(\omega_\mu + \omega_{\mu'}) \tag{26}$$

with

$$G_{\mathbf{r}}(\omega_\mu) = \frac{1}{2E_c} \cdot \sum_{[n_{\mathbf{i}}]} \frac{e^{-\frac{4}{y} \sum_{\mathbf{ij}} \frac{U_{\mathbf{ij}}}{U_{\mathbf{00}}} (n_{\mathbf{i}} + \frac{q_{\mathbf{i}}}{2e})(n_{\mathbf{j}} + \frac{q_{\mathbf{j}}}{2e})}}{1 - 4[\sum_{\mathbf{j}} \frac{U_{\mathbf{rj}}}{U_{\mathbf{00}}} (n_{\mathbf{i}} + \frac{q_{\mathbf{i}}}{2e}) - i \omega_\mu]^2} \cdot \frac{1}{Z_0}. \tag{27}$$

In Eq. (27) Z_0 is given by

$$Z_0 = \sum_{[n_{\mathbf{i}}]} e^{-\frac{4}{y} \sum_{\mathbf{ij}} \frac{U_{\mathbf{ij}}}{U_{\mathbf{00}}} (n_{\mathbf{i}} + \frac{q_{\mathbf{i}}}{2e})(n_{\mathbf{j}} + \frac{q_{\mathbf{j}}}{2e})}$$

with $U_{\mathbf{ij}} = 4e^2 C^{-1}_{\mathbf{ij}}$, $E_C = e^2 C^{-1}_{\mathbf{rr}}/2$ and $y = k_B T_c / E_C$. In terms of Matsubara frequencies the Ginzburg-Landau free energy (13) becomes [28,11,12]

$$F[\psi] = \frac{1}{\beta} \sum_{\mu \mathbf{ij}} \psi^*_{\mathbf{i}}(\omega_\mu) \left[\frac{2}{E_J} \gamma^{-1}_{\mathbf{ij}} - G_{\mathbf{i}}(\omega_\mu) \delta_{\mathbf{ij}} \right] \psi_{\mathbf{j}}(\omega_\mu) \ . \tag{28}$$

Equation (28) is the pertinent starting point for the analysis of the phase boundary between the insulating and the superconducting phases in JJA with arbitrary capacitance matrix and with a generic charge frustration.

III. MEAN FIELD RESULTS

In the following we shall illustrate the steps involved in the derivation [12] of the equation determining the phase boundary in the plane $(\alpha, K_B T_c/E_C)$, in mean-field theory and for a system with arbitrary capacitance matrix and a uniform distribution of offset charges. For this purpose, it is convenient to expand the fields $\psi_{\mathbf{i}}(\omega_\mu)$ and $G_{\mathbf{i}}(\omega_\mu)$ in terms of the vectors of the reciprocal lattice $\mathbf{k}$. One has

$$\psi_{\mathbf{i}}(\omega_\mu) = \frac{1}{N}\sum_{\mathbf{k}} \psi_{\mathbf{k}}(\omega_\mu) e^{i\mathbf{k}\cdot\mathbf{i}} \tag{29}$$

$$G_{\mathbf{i}}(\omega_\mu) = \frac{1}{N}\sum_{\mathbf{k}} G_{\mathbf{k}}(\omega_\mu) e^{i\mathbf{k}\cdot\mathbf{i}} \; . \tag{30}$$

Moreover

$$\gamma_{\mathbf{ij}}^{-1} = \frac{1}{N}\sum_{\mathbf{k}} \gamma_{\mathbf{k}}^{-1} e^{i\mathbf{k}\cdot(\mathbf{i}-\mathbf{j})}, \tag{31}$$

where $\gamma_{\mathbf{k}}^{-1}$ is the inverse of the Fourier transform of the Josephson coupling strength $\gamma_{\mathbf{ij}}$. As a consequence

$$\gamma_{\mathbf{k}}^{-1} = \frac{1}{\sum_{\mathbf{p}} e^{-i\mathbf{k}\cdot\mathbf{p}}},$$

where $\mathbf{p}$ is a vector connecting two nearest neighbors sites. Expanding in $\mathbf{k}$, one gets

$$\gamma_{\mathbf{k}}^{-1} = \frac{1}{z} + \frac{\mathbf{k}^2 a^2}{z^2} + \cdots \tag{32}$$

where a is the lattice spacing. The first term in Eq. (32) provides the mean field theory approximation which, as expected, is exact in the limit of large coordination number.

The Ginzburg-Landau free energy (28) reads

$$F[\psi] = \frac{1}{\beta N}\sum_{\mu\mathbf{k}\mathbf{k}'} \psi_{\mathbf{k}}(\omega_\mu)^* \left[\frac{2}{E_J}\gamma_{\mathbf{k}}^{-1}\delta_{\mathbf{k}\mathbf{k}'} - \frac{G_{\mathbf{k}-\mathbf{k}'}(\omega_\mu)}{N}\right] \psi_{\mathbf{k}'}(\omega_\mu). \tag{33}$$

Using Eq. (32) and keeping only terms of zero-th order in ω_μ and $\mathbf{k}$, one obtains the mean field theory approximation to the coefficient of the quadratic term of F:

$$F[\psi] \simeq \frac{1}{\beta N}\sum_{\mathbf{k}\mu}\left[\frac{2}{E_J z} - G_{\mathbf{0}}(0) + \cdots\right] |\psi_{\mathbf{k}}(\omega_\mu)|^2 \; . \tag{34}$$

The equation for the phase boundary line then reads as

$$1 = z\frac{E_J}{2} G_{\mathbf{0}}(0), \tag{35}$$

with

$$G_{\mathbf{0}}(0) = \frac{1}{N}\sum_{\mathbf{r}} G_{\mathbf{r}}(\omega_\mu = 0, T = T_c). \tag{36}$$

As evidenced in Ref. [12], the MF theory approximation amounts to neglect all the higher order terms in Eq. (33). Equation (35) determines the relation between T_c and α at the phase boundary within the MF approximation.

For a uniform distribution of offset charges Eq.(35) simplifies further since in Eq. (36) $G_{\mathbf{r}}$ does not depend on $\mathbf{r}$. As a consequence, the phase boundary equation becomes

$$1 = \alpha \cdot \sum_{[n_{\mathbf{i}}]} \frac{e^{-\frac{4}{y}\sum_{\mathbf{ij}} \frac{U_{\mathbf{ij}}}{U_{\mathbf{00}}}(n_{\mathbf{i}}+\frac{q}{2e})(n_{\mathbf{j}}+\frac{q}{2e})}}{1 - 4[\sum_{\mathbf{j}} \frac{U_{\mathbf{0j}}}{U_{\mathbf{00}}}(n_{\mathbf{i}} + \frac{q}{2e})]^2} \cdot \frac{1}{Z_0} \tag{37}$$

with

$$\alpha = \frac{zE_J}{4E_c}$$

and

$$Z_0 = \sum_{[n_{\mathbf{i}}]} e^{-\frac{4}{y}\sum_{\mathbf{ij}}\frac{U_{\mathbf{ij}}}{U_{\mathbf{00}}}(n_{\mathbf{i}}+\frac{q}{2e})(n_{\mathbf{j}}+\frac{q}{2e})}.$$

In the following we shall illustrate through examples the physical implications of Eq. (37).

A. Self Charging Model

For a diagonal capacitance matrix, $U_{\mathbf{ij}} = \delta_{\mathbf{ij}} U_{\mathbf{00}}$, one singles out only the self-interaction of plaquettes. Eq. (37) becomes

$$\frac{1}{\alpha} = g(q,y) \tag{38}$$

where

$$g(q,y) = \frac{\sum_n e^{-\frac{4}{y}(n+q/2e)^2}\frac{1}{1-4(n+q/2e)^2}}{\sum_m e^{-\frac{4}{y}(m+q/2e)^2}}. \tag{39}$$

We observe that Eq. (38) is invariant under $q \to q+2e$ and symmetric around $q = n+e$, where n is an integer:

$$g(q+2ne,y) = g(q,y); \qquad g\left[(2ne+e)+q,y\right] = g\left[(2ne+e)-q,y\right]. \tag{40}$$

In Fig. 1 we plot T_c as a function of α. One sees that there is no superconductivity for $\alpha < 1$. Due to the periodicity of Eq. (38) this holds for any integer q. For $q = e$, one gets

$$\alpha = \frac{e^{-\frac{1}{y}} + \sum_{n=1}^{+\infty} e^{-\frac{4}{y}(n+\frac{1}{2})^2}}{\frac{4+y}{4y}e^{-\frac{1}{y}} + \sum_{n=1}^{+\infty}\frac{1}{1-4(n+\frac{1}{2})^2}e^{-\frac{4}{y}(n+\frac{1}{2})^2}}. \tag{41}$$

From Fig. 2, in which we plot T_c vs. α, one sees that superconductivity is attained for all the values of α, since the superconducting order parameter at zero temperature is different from zero: a uniform offset charge $q = e$ always *favors* superconductivity. In Fig. 3 we plot the phase boundary line as a function of the charge frustration, for a finite value of the critical temperature T_c and also for $T_c \to 0$. We observe that the limit $T_c \to 0$ is singular; nevertheless, taking this limit in the finite temperature Eq. (35), one finds an expression for the phase boundary line in good agreement with the results reported in Refs. [10,11,29].

B. Nondiagonal Capacitance Matrices

In this Section we analyze the situation arising when the diagonal interaction matrix element $U_{\mathbf{00}}$ and the nearest neighbor interaction matrix element $U_{\mathbf{0p}} = \theta\, U_{\mathbf{00}}$ are nonzero. To do this, one should expand the critical line Eq. (37) for $q = 0$ and small critical temperatures [30]:

$$\alpha = 1 + \big[\frac{8}{3} + 2z(1 - \frac{1}{1-4\theta^2})\big]e^{-\frac{4}{y}} + \ldots \quad .$$

When $\theta > 1/\sqrt{4+3z}$, the coefficient of the exponential $e^{-4/y}$ is negative and the phase boundary line $\alpha = \alpha(T_c)$ first bends to the left; when the critical temperature is high enough, it bends to the right, favoring the insulating phase. This is an indication for the possibility of observing *reentrant superconductivity* in these systems: fixing the parameter of the JJA (i.e., fixing α), by decreasing the temperature it is possible to go down from an insulating state to a superconducting one and then, further decreasing T_c, to go back to the insulator.

As evidenced by Fishman and Stroud [30], the regime of physical interest is $\theta < 1/z$; namely, when the capacitance matrix is invertible. Therefore, in dimensions $D \geq 2$, reentrance shall not occur with $q = 0$ and nearest neighbor interaction matrix [30]. With $q = e$ and a nearest neighbor interaction matrix, the situation is different [12]. We define

$$E[n_{\mathbf{i}}] = \sum_{\mathbf{i}\,\mathbf{j}} \frac{U_{\mathbf{ij}}}{U_{\mathbf{00}}}(n_{\mathbf{i}} + \frac{1}{2})(n_{\mathbf{j}} + \frac{1}{2}) \tag{42}$$

the energy of a generic charge distribution on the lattice $\{n_{\mathbf{i}}\}$. Denoting with $n^0_{\mathbf{i}}$ and $n^1_{\mathbf{i}}$ the charge distributions of the two lowest lying energy states and with E^0 and E^1 the corresponding energies, the low temperature expansion of Eq. (37) yields

$$\alpha = \frac{\sum_{[n^0]} e^{-\frac{4}{y}E^0} + \sum_{[n^1]} e^{-\frac{4}{y}E^1} + \cdots}{\sum_{[n^0]} \frac{e^{-\frac{4}{y}E^0}}{1-4\left[\sum_{\mathbf{j}} \frac{U_{\mathbf{0j}}}{U_{\mathbf{00}}}(n^0_{\mathbf{j}}+\frac{1}{2})\right]^2} + \sum_{[n^1]} \frac{e^{-\frac{4}{y}E^1}}{1-4\left[\sum_{\mathbf{j}} \frac{U_{\mathbf{0j}}}{U_{\mathbf{00}}}(n^1_{\mathbf{j}}+\frac{1}{2})\right]^2} + \cdots} . \tag{43}$$

Independently on the explicit form of $U_{\mathbf{ij}}$, $E[n_{\mathbf{i}}]$ (for a square lattice in D dimensions) reaches its minimum value when $(n^0_{\mathbf{i}} + \frac{1}{2}) = \pm\frac{1}{2}(-1)^{i_1+i_2+\ldots+i_D}$ with i_j $(j = 1, ..., D)$ the components of the lattice position vector $\mathbf{i}$ in units of the lattice spacing.

For models with nearest-neighbor interaction, i.e., $U_{\mathbf{ij}}/U_{\mathbf{00}} = \delta_{\mathbf{ij}} + \theta \sum_{\mathbf{p}} \delta_{\mathbf{i+p},\mathbf{j}}$ with $\sum_{\mathbf{p}}$ denoting summation over nearest neighbors, the first excited state has an energy $E[n^1_{\mathbf{i}}] = E[n^0_{\mathbf{i}}] + z\theta$, where $E[n^0_{\mathbf{i}}]$, the ground state energy, is given by $\sum_{\mathbf{i}}(1 - z\theta)/4$.

With the above values of $E[n^0_{\mathbf{i}}]$ and $E[n^1_{\mathbf{i}}]$ and keeping only the leading order term in T_c, Eq. (43) becomes

$$\alpha = (1 - (1 - z\theta)^2) \cdot (1 + a_1 e^{-\frac{4}{y}z\theta} + \cdots) \tag{44}$$

where

$$a_1 \equiv \left(1 - \frac{1 - (1 - z\theta)^2}{1 - (1 + z\theta)^2}\right) + z\left(1 - \frac{1 - (1 - z\theta)^2}{1 - (1 - (z-2)\theta)^2}\right). \tag{45}$$

Reentrant behavior at low temperature occurs when the coefficient of the exponential is negative, namely when $a_1 < 0$. In Fig. 4 we plot T_c versus α for $\theta = 0.05$ and $z = 6$. The resulting diagram exhibits reentrance in the insulating phase even for models with nearest neighbors interaction. For a detailed study of the lobe diagrams in JJA with non-diagonal capacitance matrices, see [1].

IV. CAPACITIVE DISORDER

In this Section, we shall determine the finite temperature phase diagram of JJA with capacitive disorder (i.e., with random offset charges and/or random self-capacitances). To derive the phase boundary between the insulating and the superconducting phase, we shall use the path-integral approach for quantum JJA with offset charges and general capacitance matrices reviewed in the previous Sections. We find that charge disorder supports superconductivity and that the relative variations of the insulating and superconducting regions depend on the mean value q of the charge probability distribution: when $q = 0$, increasing the disorder leads to an enlargement of the superconducting phase. If the charge disorder is sufficiently strong ($\sigma \gtrsim e$), the lobe structure [1] disappears: in other words, the phase boundary line (and the correlation functions) do not depend any longer on q. In the following, we shall provide a quantitative analysis of this phenomenon. Also the randomness of the self-capacitances leads to remarkable effects, namely, the superconducting phase increases with respect to the case where disorder is not present.

We shall consider several probability distributions which we expect to provide a realistic description of experimental situations. The low temperature behavior obtained by a pertinent extrapolation of our finite T results is consistent with the phase diagram obtained in Ref. [13].

For a given realization of the disorder, the Ginzburg-Landau free energy (i.e., the free energy near the transition) is given by Eq. (34). We shall perform a quenched average, in which each of the random variables takes a unique value as the statistical variables fluctuate. This corresponds to taking the average of the logarithm of the partition function, i.e., the free energy. The average of the free energy over all the possible realization of the disorder allows for the evaluation of the effect of a random charge frustration $\{q_{\mathbf{i}}\}$ or a random diagonal charging energy terms $U_{\mathbf{ii}} = 4e^2 C^{-1}_{\mathbf{ii}}$. The pertinent starting point for the analysis of these situations is then

$$\bar{F}[\psi] = \int d\{X\}P(\{X\})F[\psi] \tag{46}$$

where $P(\{X\})$ is a given probability distribution and $d\{X\}P(\{X\}) = \prod_{\mathbf{i}} dq_{\mathbf{i}}P(q_{\mathbf{i}})$ if one analyzes the effect of random offset charges or $d\{X\}P(\{X\}) = \prod_{\mathbf{i}} dU_{\mathbf{ii}}P(U_{\mathbf{ii}})$ for random charging energies. The random variables on different sites are taken to be independent. The phase boundary line between the insulating and the superconducting phase is determined by requiring that $\bar{F} = 0$, which in turn leads to [15]

$$1 = z\frac{E_J}{2}\bar{G}_{\mathbf{0}}. \tag{47}$$

A. Random offset charges

In the following, we shall consider three different random offset charges probability distribution with mean q and width σ. That is, a Gaussian distribution $P(q_{\mathbf{i}}) = const \cdot e^{-(q_{\mathbf{i}}-q)^2/2\sigma^2}$, a uniform distribution $P(q_{\mathbf{i}}) = const$ between $q-\sigma$ and $q+\sigma$ and 0 otherwise, and a sum of δ-like distributions $P(q_{\mathbf{i}}) = \sum_n p_n\delta(q_{\mathbf{i}} - ne)$, with $\sum_n p_n = 1$. For a diagonal capacitance matrix, Eq. (47) leads to

$$\frac{1}{\alpha} = \int dqP(q)g(q,y) \tag{48}$$

with $\alpha = zE_J/4E_c$, $y = k_BT_c/E_c$ and $g(q,y)$ given by Eq. (39). If one considers the infinite-range hopping limit, one still gets Eq. (48) [15].

The results obtained from Eq. (48) with a Gaussian distribution are displayed in Fig. 5. One observes that, when $q = 0$, increasing σ favors the superconducting phase while, when $q = e$, increasing σ leads to the increase of the insulating phase. For large σ (i.e. $\sigma \gtrsim e$), the phase boundary line is the same for all the values of q (in Fig. 5 the large σ behavior is represented by the bold line). This is expected since, when σ is large, the average free energy $\bar{F}$ does not depend any longer on q.

As seen in the previous Sections an useful representation of the phase diagram is provided if one plots, at fixed critical temperature, the phase boundary line on the plane $q-\alpha$. Without disorder one observes the lobe structure as discussed in Section III. In the presence of weak disorder and for $T_c \to 0$, the lobes shrink, evidencing a decrease of the insulating phase: for a Gaussian (or unbounded) distribution the insulating phase completely disappears even for an arbitrarily weak disorder [13].

In Fig. 6 we plot the phase boundary line on the plane $q-\alpha$ at finite T_c for the Gaussian and uniform distributions [15]. When the disorder increases the lobes flatten even at finite temperature and the same lobe structure is obtained from both distributions. In the limit $T_c \to 0$, one recovers the result of Ref. [13]. This can be easily seen if one observes that, at very low temperatures, for $|q| < e$, one has from Eq. (39)

$$g(q, y \to 0) = \frac{1}{1 - 4(\frac{q}{2e})^2}. \tag{49}$$

Without disorder ($\sigma = 0$), Eq. (48) simply gives $\alpha = 1 - 4(q/2e)^2$. With the Gaussian distribution, since g has a pole in the half-integer value of the Cooper charge, the integral in Eq. (48) diverges and $\alpha \to 0$, i.e., the lobes disappear for every value of σ. As evidenced in Fig. 7, for a uniform distribution, when $\sigma > e$, then $\alpha \to 0$; when $\sigma < e$, $\alpha \to 0$ only for $e - \sigma \le q \le e + \sigma$ in agreement with [13].

Another interesting situation arises if one considers

$$P(q_{\mathbf{i}}) = \sum_n p_n\delta(q_{\mathbf{i}} - ne), \tag{50}$$

with $\sum_n p_n = 1$. This corresponds to a random distribution of charges which are integer multiples of e and, actually, this is the most realistic situation for a random distribution. In fact, the probability distributions employed before should be viewed as fictitious continuous distributions, i.e., the properties of the overall distribution of charges (mean value and width) can be well approximated with a continuous distribution $P(q)$. Inserting the probability distribution (50) in Eq. (48) one has

$$\frac{1}{\alpha} = \int dq \sum_n p_n\delta(q - ne)g(q,y) = \sum_{odd} p_n g(ne,y) + \sum_{even} p_n g(ne,y),$$

where $\sum_{odd}$ ($\sum_{even}$) is a sum restricted to odd (even) integer. From Eq. (40), one has $g(2ne, y) = g(0, y)$ and $g((2n+1)e, y) = g(e, y)$, which leads to

$$\frac{1}{\alpha} = p_0\, g(0, y) + p_e\, g(e, y), \tag{51}$$

where $p_0 = \sum_{even} p_n$ ($p_e = \sum_{odd} p_n$) is the probability that the offset charge q is an even (odd) integer multiple of e. In Fig. 8 we plot the phase boundary line (51) for $p_0 = p_e = 1/2$.

We discuss now the effects induced by nondiagonal capacitance matrices in the limit $T_c \to 0$ [15]. The phase diagram without disorder becomes richer [1]; for concreteness, we shall consider on-site and a weaker nearest-neighbor (NN) interaction, i.e., the inverse capacitance matrix is restricted to diagonal and NN terms. If one defines θ as the ratio between NN and diagonal terms, one should restrict only to $z\theta < 1$ in order to insure the invertibility of the capacitance matrix [30]. Without disorder, at very low temperatures an insulating lobe around $q = e$ appears [11]: the width of this lobe is $z\theta/(1 + z\theta)$. Putting $W = 1 + z\theta$, Eq. (39) for $|q/2e| < 1/2W$ gives $g(q, y \to 0) = 1/[1 - 4W^2(q/2e)^2]$; for $1/2W < q/2e < 1 - (1/2W)$ it becomes,

$$g(q, y \to 0) = -\frac{1}{2}\left[\frac{1}{(2W\frac{q}{2e} - 1)(2W\frac{q}{2e} - 3)} + \frac{1}{(2W(\frac{q}{2e} - 1) + 1)(2W(\frac{q}{2e} - 1) + 3)}\right]. \tag{52}$$

In presence of disorder, Eq. (48) for a uniform distribution gives $\alpha = 0$ for $(1/2W) - \sigma \leq q \leq (1/2W) + \sigma$ and $1 - (1/2W) - \sigma \leq q \leq 1 - (1/2W) + \sigma$. Thus, the lobe width decreases as $(z\theta - 2\sigma W)/W$. One sees that for $\sigma = z\theta/2W$ the insulating lobe around $q = e$ disappears. This phenomenon is evidenced in Fig. 9.

B. Random Self-Capacitances

In many instances, it may happen that the network's parameters are not uniform across the whole array: despite recent advances in fabrication techniques, with the use of submicron lithography, variation of junction parameters associated to the shape of the islands can be also of 20% [1]. Thus, it is relevant in many practical situations to study JJA with randomly distributed self-capacitances: this corresponds to have a random diagonal charging energy [15,31]. In Ref. [31] the effects of disorder on the spectrum of elementary excitations at low temperatures have been studied.

In this Section we shall study JJA at finite temperature with uniform charge frustration q and random self-capacitance $C_{\mathbf{ii}}$, ignoring non-diagonal contributions to the capacitance matrix. The diagonal charging energy terms $U_{\mathbf{ii}}$ are related to the self-capacitances $C_{\mathbf{ii}}$ via $U_{\mathbf{ii}} = 4e^2 C_{\mathbf{ii}}^{-1}$ and are assumed to be independently distributed according to the probability distribution $P(U_{\mathbf{ii}}) \propto e^{-(U_{\mathbf{ii}} - U_0)^2/2\sigma^2}$. The average charging energy is defined as $E_C^0 = U_0/8$. The diagonal electrostatic contribution to the energy $U_{\mathbf{ii}}$ needs to be positive.

By averaging the free energy (46), the equation for the phase boundary becomes [15]

$$\frac{1}{\alpha} = \int_0^\infty dU \frac{P(U)}{U} g(U, y) \tag{53}$$

where now $\alpha = zE_J/4E_C^0$, $y = k_B T_c/E_C^0$, and $U = U_{ii}/U_0$; the function $g(U, y)$ is given by

$$g(U, y) = \frac{\sum_n e^{-\frac{4}{y}U(n+q/2e)^2} \frac{1}{1-4(n+q/2e)^2}}{\sum_m e^{-\frac{4}{y}U(m+q/2e)^2}}. \tag{54}$$

The results of Eq. (53) are summarized in Figs. 10 and 11: when σ is small, the superconducting phase increases in comparison to the situation in which all self-capacitances are equal: this is due to the factor $1/U$ in Eq. (53), which makes larger the contribution of junctions with charging energies less than U_0. An interesting observation is that, when $q = e$ (maximum frustration induced by the external offset charges), the randomness does not modify considerably the phase diagram. This should be compared with the nonfrustrated case ($q = 0$), where randomness sensibly affects the phase diagram.

The increase of the superconducting phase is due to a decrease of the effective value of the charging energy. This behavior occurs until σ reaches a critical value (depending on the charge frustration and on the temperature), of order U_0: at this value of σ the insulating region starts to increase. This is due to the asymmetry of the distribution, which has its peak in U_0, but only for positive values. This phenomenon is present also if one considers different distributions and is clearly seen in Fig. 12, where the phase diagram in the q-α plane for different values of the variance σ is plotted.

V. CONCLUDING REMARKS

In this paper, we reviewed the use of the path integral approach to finite temperature mean-field theory to analyze the effects induced by offset charges on the finite temperature phase diagram of Josephson junction arrays. We provided, for a general Coulomb interaction matrix, the explicit derivation of the equation for the phase boundary line between the insulating and superconducting phase.

The resulting phase diagram is drawn in the diagonal case for a generic uniform offset charge distribution q: with $q = e$, the superconducting phase increases with respect to $q = 0$, and the model exhibits superconductivity for all the values of $\alpha = zE_J/4E_C$. An offset charge $q = e$ tends to decrease the charging energy and thus favors the superconducting behavior even for small Josephson energies.

For the model with nearest neighbor inverse capacitance matrix and uniform offset charge $q = e$, we determined, in the low temperature expansion, the most relevant contributions to the equation for the phase boundary. For this purpose we explicitly constructed the charge distributions on the lattice corresponding to the lowest energies: a reentrant behavior is found even with a short ranged interaction.

We also obtained the phase diagram at finite temperature of JJA with capacitive disorder. For a random distribution of offset charges with mean q and variance σ, one has that for $\sigma \gtrsim e$, the phase boundary line coincides for any value of q and the lobe structure on the plane $q - \alpha$ disappears (α is the ratio between the Josephson and charging energies). For very low temperatures there is agreement with the result of Ref. [13].

If one considers also a nearest-neighbor interaction, the insulating lobe around $q = e$, which arises in absence of disorder, is destroyed even for small values of σ. For arrays with random charging energies, when the variance of the probability distribution is smaller than a critical value, the superconducting phase increases with respect to the situation in which all self-capacitances are equal.

It is comforting to observe that the finite temperature MF theory approach developed in this paper provides results which are in good agreement with those obtained by recent quantum Monte Carlo simulations [32] and by use of improved variational methods [33].

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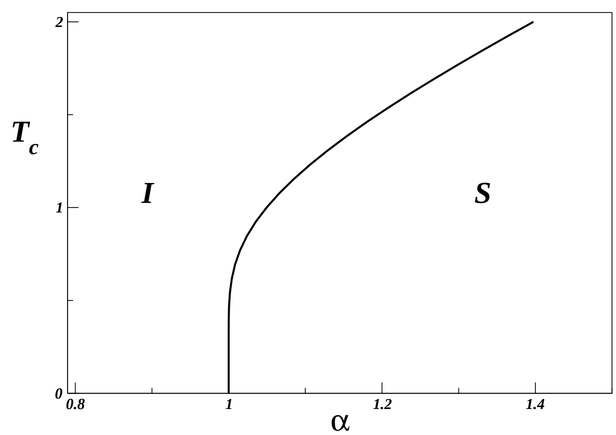

FIG. 1. Phase diagram for the diagonal model without charge frustration. The critical temperature T_c is in units of k_B/E_C and α stands for the ratio $zE_J/4E_C$. The **I** and **S** indicate, respectively, insulating and superconducting phase.

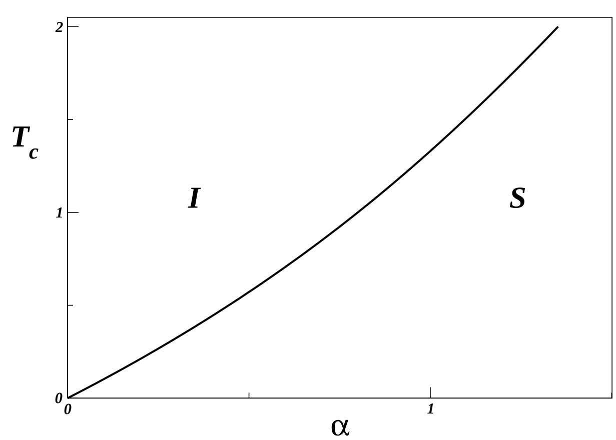

FIG. 2. Phase diagram of the diagonal model with half-integer charge frustration $q = e$; T_c is in units of k_B/E_C.

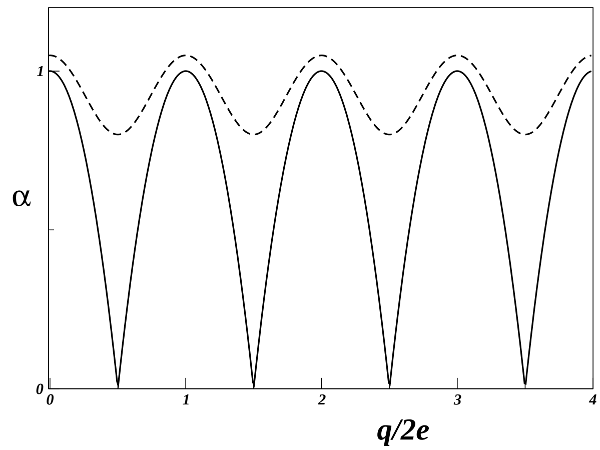

FIG. 3. Lobe diagram for $T_c \to 0$ (solid line) and $T_c = k_B/E_C$ (dashed line). As the Hamiltonian (1) is periodic in q with period $2e$, the lobes are repeated on the q axis.

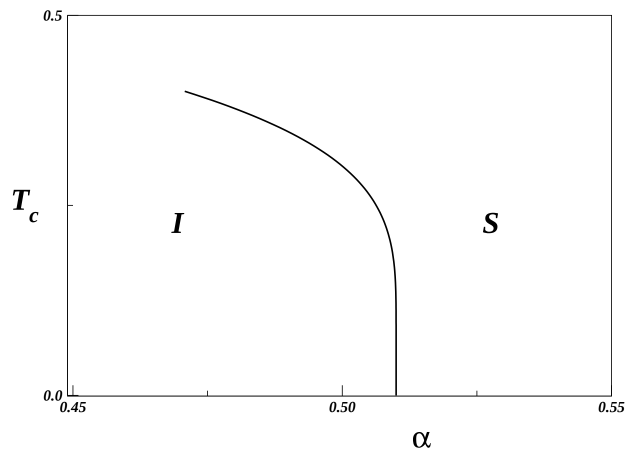

FIG. 4. Low temperature expansion of the critical line with a short-ranged inverse capacitance matrix. In the plot, $z = 6$ and $\theta = 0.05$, where θ is the ratio between nearest-neighbor and diagonal terms.

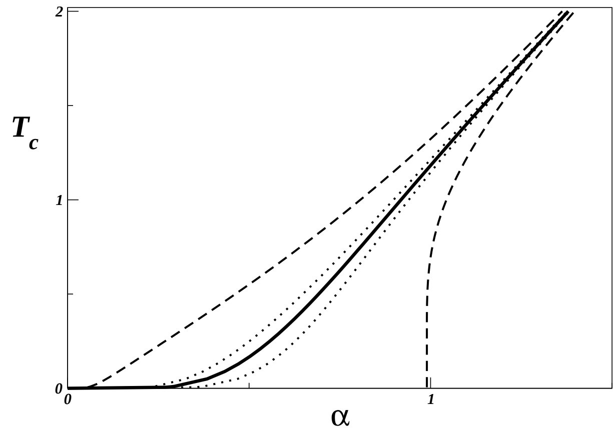

FIG. 5. Critical line for random offset charges with Gaussian distribution (T_c is in units of k_B/E_C). The bold line is for $\sigma = e$: for $\sigma \gtrsim e$ no significant deviations from this line are observed for all the values of q. To the left (right), we plot $q = e$ ($q = 0$); in the plot $\sigma/2e = 0.05$ (dashed lines) and 0.3 (dotted lines).

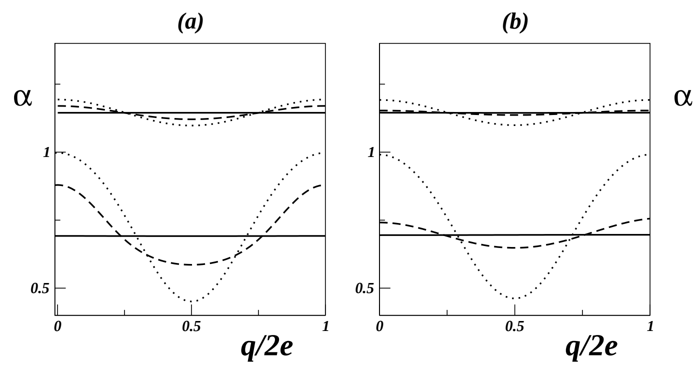

FIG. 6. Phase diagram with diagonal capacitances and random offset charges with uniform (a) and Gaussian (b) distribution. Top (bottom) of the figures: $k_BT/E_C = 1.5(0.5)$. We plot the cases $\sigma/2e = 0.05$ (dotted lines), 0.3 (dashed lines)and 0.5 (solid lines). For large $\sigma/2e$ the phase boundary line is flat and it is the same for both distributions.

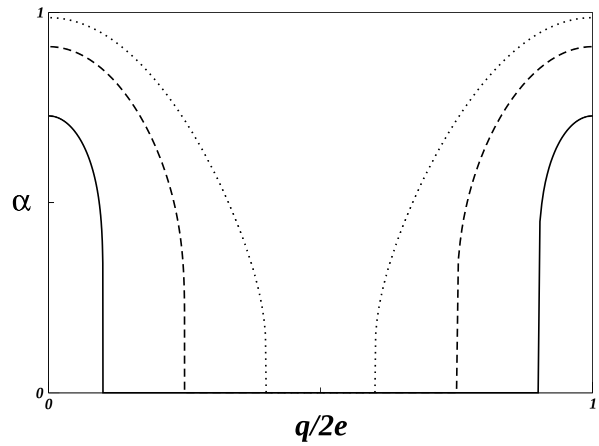

FIG. 7. Phase diagram at very low critical temperatures for a diagonal inverse capacitance matrix and random offset charges with uniform distribution. We plot $\sigma/2e = 0.1$ (dotted line), 0.25 (dashed line) and 0.40 (solid line).

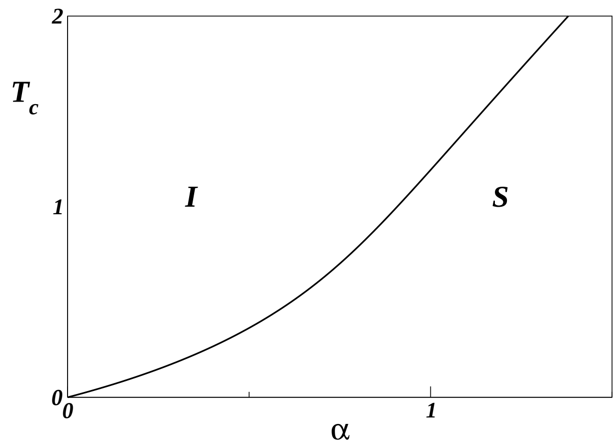

FIG. 8. Phase diagram for random offset charges with the probability distribution given by Eq. (50) and diagonal capacitance matrix. In the plot we take $p_0 = p_e = 1/2$.

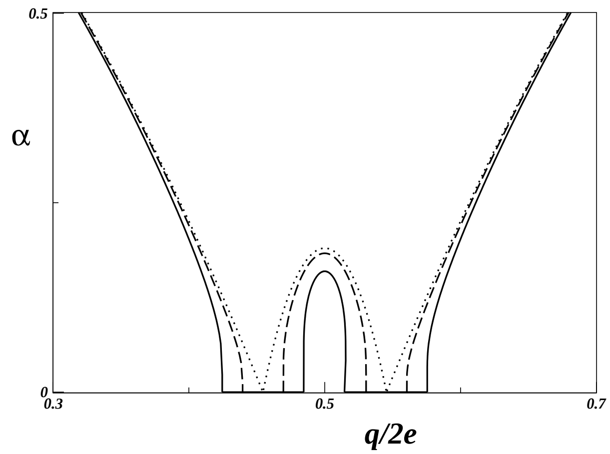

FIG. 9. Disappearance of the insulating lobe around $q = e$ for a short-ranged inverse capacitance matrix. The phase diagram is plotted for $T_c \to 0$ and with random offset charges uniformly distributed. In the plot $z\theta$ is equal to 0.1 while $\sigma/2e$ is respectively 0 (dotted line), 0.015 (dashed line) and 0.03 (solid line). For this value of $z\theta$, the lobe disappears at $\sigma/2e = 0.045$.

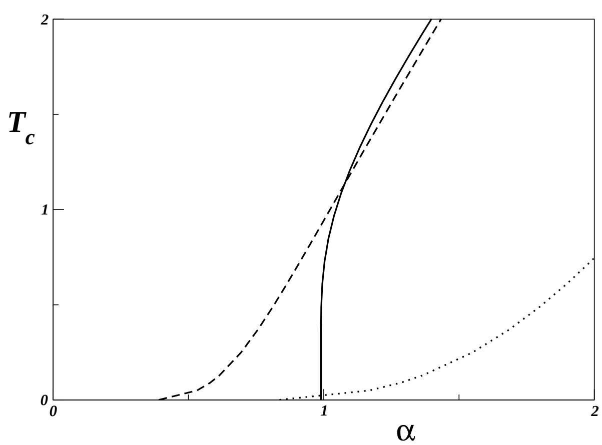

FIG. 10. Phase diagram in the $T_c - \alpha$ plane for random diagonal capacitance with Gaussian distribution and without charge frustration (T_c is in units of k_B/U_0) while $\sigma/2e$ is 0.1 (solid line), 1 (dashed line) and 5 (dotted line). For each line, the superconducting (insulating) phase lies on the right (left).

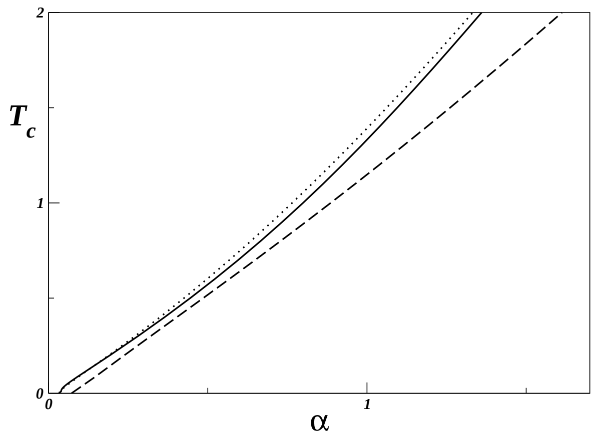

FIG. 11. Phase diagram in the $T_c - \alpha$ plane for random diagonal capacitance with Gaussian distribution and uniform offset charge $q = e$ (T_c is in units of k_B/U_0) while $\sigma/2e$ is 0.1 (solid line), 1 (dashed line) and 5 (dotted line).

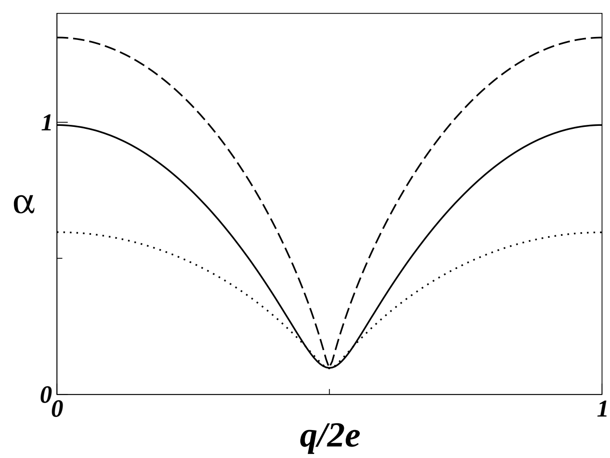

FIG. 12. Phase diagram in the $q - \alpha$ plane for random diagonal capacitance with Gaussian distribution and uniform offset charge at $k_B T/U_0 = 0.1$. We plot $\sigma/2e = 0.1$ (solid line), 1 (dotted line) and 5 (dashed line).